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LETTER TO THE EDITOR

Critical dynamics at the percolation threshold

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Abstract. Dilution-induced criticality is discussed using exact and scaling descriptions. A simple exact argument is used to obtain the scaling form for the density of states in dilute chains.

For higher-dimensional systems it is shown that the condition for the existence of a 'fracton' edge is equivalent to the violation of a scaling relation between the fracton dimension of the infinite cluster, $\tilde{d}$ and various static exponents. In addition the result $\tilde{d}d/d_f$ is obtained for the fracton dimensionality of the finite clusters near the percolation threshold; d_f is the fractal dimension of the infinite cluster.

In this letter we discuss dilution-induced criticality in random systems with a Goldstone symmetry. Recent years have seen the emergence of scaling descriptions of excitations near the percolation threshold (Korenbit and Shender 1978, Shender 1976, 1978) and in particular of geometrical viewpoints and recursive approaches to dynamics on fractals such as the percolating network (Kirkpatrick 1973, 1979, Stinchcombe 1983a, b, 1984, Domany *et al* 1983, Harris and Stinchcombe 1983). The associated crossover phenomena have received much attention lately. In particular it is known that the modes involved in the critical dynamics cross over from hydrodynamic spin waves at low frequency (Harris and Kirkpatrick 1977) to new excitations ('fractons') above a frequency such that the wavelength is approximately the (percolative) correlation length, ξ. This implies that the density of states will be anomalous at the critical concentration $p_c(\xi=\infty)$ and it has in fact been shown to diverge for ferromagnetic spin waves at the percolation threshold in two dimensions (Lewis and Stinchcombe 1984, Lewis and O'Brien 1984). Such behaviour can be characterised by a new (fracton) dimension for the infinite cluster, in addition to the fractal dimension which relates to the scaling of its 'mass' (Mandelbrot 1977). The crossover can give rise to anomalous effects, such as those occurring in diffusion at p_c (Alexander and Orbach 1982, Rammal and Toulouse 1982a, b, Pandey and Stauffer 1983, Pandey *et al* 1984). Another effect which has been suggested and could explain anomalies in the thermal conduction and specific heat of certain glasses is the existence of a sharp edge (the fracton edge) in the density of states at the crossover frequency (Rosenberg 1984, Orbach 1985, Derrida *et al* 1984, Entin-Wohlman *et al* 1984).

In this letter we consider a number of points related to the above picture. We give an exact scaling description of the density of states in one-dimensional diluted systems and then consider higher-dimensional systems where the existence of the fracton edge is discussed from the scaling viewpoint. Finally we point out that dyanamic experiments

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performed on a percolating lattice (for instance, dilute magnetic alloys) would not measure the fracton dimension of the infinite cluster but instead a related exponent which is determined by the large but finite clusters which are present. This exponent is calculated.

Though our conclusions readily generalise to phonon and diffusion problems, to be specific we consider low temperature spin waves in the bond diluted Heisenberg ferromagnet at concentration p in which the exchange couplings J are independently distributed random variables with the distribution

$$P(J)=(1-p)\delta(J)+p\delta(J-1). \tag{1}$$

In one dimension the fact that any dilution breaks the system into independent chain segments has already been used (Odagaki and Lax 1980, Stinchcombe and Harris 1983, Maggs and Stinchcombe 1984) to calculate the full dynamic response function, χ, for the bond diluted chain as a function of wavevector k, frequency ω and the static percolative correlation length ξ. In the critical regime where k, ω and $\xi^{-1}\to 0$ but $k\xi$ and ωk^{-z} remain finite, χ exhibits dynamic scaling (Halperin and Hohenberg 1969) i.e.

$$\chi(k,\omega,\xi)=k^{2-\eta+z}F(k\xi,\omega k^{-z}). \tag{2}$$

From χ we can find the related instantaneous or $t=0$ correlation function, $\bar{\chi}$, by integration over ω and the density of states by integration over k. We see that $\bar{\chi}$ and ρ are homogeneous functions of their remaining variables. We shall now give a simple argument to show how this occurs for $\rho(\omega)$ while at the same time obtaining the scaling function involved.

We are interested in the critical regime where the static percolative correlation length $\xi=|\ln p|^{-1}$ diverges. This distance is a measure of the typical length of the segments into which a chain breaks up when randomly diluted. We shall need to know the distribution of chain lengths in the system and the low lying eigenfrequencies of the long chains which dominate $\rho(\omega)$ for diverging ξ. The probability distribution of segment lengths N in a dilute chain is $P(N)=p^N(1-p)$ and for a long segment with free boundary conditions the eigenfrequencies of the low energy modes are $\omega=(m\pi/(N+1))^2$ (where m is integral). We introduce the integrated density of states, $G(\omega,\xi)=\int^{\omega}\rho(\omega')\,d\omega'$ to count the total number of states in the system with energy less than or equal to ω. For a chain segment to contribute to $G(\omega,\xi)$, it must be of sufficient length for there to be at least one mode of frequency less than or equal to ω; that is, the length of the segment must satisfy the inequality $N\geqslant\pi/\sqrt{\omega}-1\equiv n_1(\omega)$. If we choose ω so that n_1 is integral we find the total number of segments of at least this length to be p^{n_1}. In the same way we can find the number of segments with at least m modes contributing. Then the total number of states contributing to $G(\omega,\xi)$ is given by a geometric series over m which can be summed to give

$$G(\omega,\xi)=\frac{p^{\pi/\sqrt{\omega}-1}}{1-p^{\pi/\sqrt{\omega}}}\simeq\frac{\exp(-\pi/\xi\sqrt{\omega})}{1-\exp(-\pi/\xi\sqrt{\omega})}. \tag{3}$$

As $\xi\to\infty$ with $\xi^2\omega$ constant, $G(\omega,\xi)$ is defined for all finite values of the argument of equation (3). The density of states is now found by simple differentiation. It is easy to see that it will be of the form

$$\rho(\omega)=\omega^a K(\omega\xi^b) \tag{4}$$

where $a=-1$ and $b=2$. A similar result applies for the diluted diffusion problem (see Gefen *et al* 1983) and the diluted phonon system where $a=-1$, $b=1$. Similar simple arguments can be used to calculate other properties of dilute systems in one dimension.

We now turn to the dynamics of dilute lattices in two or more dimensions and in particular to a consideration of the fracton edge. The argument will here be given again in terms of excitations in a magnetic system but the argument for phonons is only trivially different.

For $p > p_c$ and large ξ crossover arguments suggest that there are two distinct scaling regimes separated by a crossover frequency $\omega_c \cong D/\xi^2$ which divides the low energy extended states and the higher energy localised states which for the vibrational case have been called fractons. It has been conjectured (Derrida *et al* 1984, Entin-Wohlman *et al* 1984) that the density of states has a ('fracton') edge at ω_c and our aim now is to see whether this feature is required by scaling. In the above expression for ω_c, D is the spin wave stiffness in the dilute lattice which is well defined for $p > p_c$. For $\omega \to 0$ the standard mode counting arguments are valid so that $\rho(\omega) = \omega^{d/2-1}/D^{d/2}$. For higher frequencies, up to the scaling limit ω_s (some small ξ-independent fraction of the maximum frequency of the pure model), the following form applies, where the power defines the fracton dimension $\tilde{d}$

$$\rho(\omega) = B\omega^{\tilde{d}/2-1}. \tag{5}$$

If we match the densities of states at ω_c by assuming that no fracton edge occurs we are able to calculate B. But there is another way of calculating B which is to demand that the integral of $\rho(\omega)$ is equal to the number of states on the infinite cluster, $P(p)$. As $p \to p_c$ the 'scaling' $(0 < \omega < \omega_s)$ and 'non-scaling' $(\omega_s < \omega)$ contributions to this integral remain of comparable order of magnitude; however, the dominant scaling contribution comes from the range $\omega_c < \omega < \omega_s$, i.e. from the magnetic equivalent of the fractons which results in $\int \rho(\omega)\,d\omega = O(B)$. Substituting the usual power law behaviour for D and ξ ($D \propto (p-p_c)^{t-\beta}$ and $\xi \propto (p-p_c)^{-\nu}$) leads to

$$\tilde{d} = 2(d\nu - \beta)/(t - \beta + 2\nu). \tag{6}$$

This relationship between the exponents would have to be satisfied for the fracton edge not to occur. It turns out however that the relation is quite well obeyed, and indeed crossover arguments have been given (Alexander and Orbach 1982, Rammal and Toulouse 1982a, b) which yield equation (6). If the relationship is indeed satisfied the existence of the fracton edge is unnecessary. The same conclusion has also been reached by Aharony *et al* (1985).

So far we have discussed the infinite cluster but at $p = p_c$ the fraction of sites in the infinite cluster is zero. The response and the density of states at low frequencies will be dominated by large but finite clusters. There is a simple relationship between the fracton dimensionality of the infinite cluster and that of the ensemble of large but finite clusters generated by the bond dilution. The argument is based on the premise that on length scales which are small compared to the size of a given finite cluster the properties of that cluster are identical to the infinite cluster.

At $p = p_c$ the distribution of the number of clusters, n_s, containing s sites is given by a power law (Stauffer 1979, 1985, Essam 1980)

$$n_s \propto s^{-2-1/\delta}. \tag{7}$$

Following our premise, on a cluster containing s sites the number of states with energy $\leqslant \omega_0$ will follow the same power law behaviour as on the infinite cluster, provided the energy is not too low for the influence of boundary conditions to be important. For any finite cluster, finite size effects will result in a low energy cut-off.

For the whole system the number of eigenfrequencies $\leq \omega_0$ is given by the weighted sum over all cluster sizes of the same power law behaviour assumed for the infinite cluster multiplied by a cut-off function to take the finite size effects into account:

$$N(\omega_0) = \sum_{s>s_0} N_s(\omega_0) n_s \tag{8}$$

with

$$N_s(\omega_0) = s\omega_0^{\bar{d}/2}. \tag{9}$$

The minimum size of cluster included in the sum for any particular ω_0 is given by the following argument: by the definition of the fractal dimension the number of sites in a cluster of linear dimensions L is $s \sim L^{d_f}$. The lower cut-off frequency depends on the cluster size in the following way:

$$\omega \sim L^{-1/\theta} \tag{10}$$

where θ is the anomalous diffusion exponent on the infinite cluster. Thus writing the cut-off size s_0 in terms of ω_0 and letting the sum (8) tend to an integral gives the following expression for the integrated density of states:

$$N(\omega_0) \sim \omega_0^{\bar{d}/2+\theta d_f/\delta}. \tag{11}$$

Using the results $\theta = \tilde{d}/2d_f$ (Alexander and Orbach 1982) and $d/d_f = 1 + 1/\delta$ we find that the fracton dimensionality of the finite clusters is

$$\bar{d} = \tilde{d}d/d_f \tag{12}$$

agreeing with Alexander (1983), Alexander *et al* (1983) and Aharony *et al* (1985).

This relationship has important consequences for the critical dynamics of real systems as this is the exponent which would be measured in most experiments on dilute ferromagnets at the percolation threshold. However the difference between the infinite cluster and finite cluster effects would be small as d_f/d is near to one for two- and three-dimensional lattices at p_c.

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